CIII. *The Invention of a General Method for determining the Sum of every 2d, 3d, 4th, or 5th, &c. Term of a Series, taken in order; the Sum of the whole Series being known.* By Thomas Simpſon, *F. R. S.*

Read Nov. 16, 1758.

AS the doctrine of Series' is of very great uſe in the higher branches of the mathematics, and their application to nature, every attempt tending to extend that doctrine may juſtly merit ſome degree of regard. The ſubject of the paper, which I have now the honour to lay before the Society, will be found an improvement of ſome conſequence in that part of ſcience. And how far the buſineſs of finding fluents may, in ſome caſes, be facilitated thereby, will appear from the examples ſubjoined, in illuſtration of the general method here delivered.

The ſeries propounded, whoſe ſum (S) is ſuppoſed to be given (either in algebraic terms, or by the meaſures of angles and ratio's, *&c.*) I ſhall here repreſent by $a + bx + cx^2 + dx^3 + ex^4$, &c. and ſhall firſt give the ſolution of that caſe, where every third term is required to be taken, or where the ſeries to be ſummed is $a + dx^3 + gx^6 + kx^6$, &c. By means whereof, the general method of proceeding, and the reſolution of every other caſe, will appear evident.

Here, then, every *third* term being required to be taken, let the ſeries ($a + dx^3 + gx^6$, &c.), whoſe value

value is sought, be conceived to be composed of *three* others.

$$\tfrac{1}{3}\times\overline{a+b\times\overline{px}+c\times\overline{px}^2+d\times\overline{px}^3+e\times\overline{px}^4,\ \&c.}$$
$$\tfrac{1}{3}\times\overline{a+b\times\overline{qx}+c\times\overline{qx}^2+d\times\overline{qx}^3+e\times\overline{qx}^4,\ \&c.}$$
$$\tfrac{1}{3}\times\overline{a+b\times\overline{rx}+c\times\overline{rx}^2+d\times\overline{rx}^3+e\times\overline{rx}^4,\ \&c.}$$

having all the *same form*, and the *same coefficients* with the series first proposed, and wherein the converging quantities px, qx, rx, are also in a determinate (tho' yet unknown) ratio to the original converging quantity x. Now, in order to determine the quantities of these ratios, or the values of p, q, and r, let the terms containing the same powers of x, in the two equal values, be equated in the common way:

So shall,

$$\tfrac{1}{3}b\times px+\tfrac{1}{3}b\times qx+\tfrac{1}{3}b\times rx=0$$
$$\tfrac{1}{3}c\times p^2x^2+\tfrac{1}{3}c\times q^2x^2+\tfrac{1}{3}c\times r^2x^2=0$$
$$\tfrac{1}{3}d\times p^3x^3+\tfrac{1}{3}d\times q^3x^3+\tfrac{1}{3}d\times r^3x^3=dx^3$$
$$\tfrac{1}{3}e\times p^4x^4+\tfrac{1}{3}e\times q^4x^4+\tfrac{1}{3}e\times r^4x^4=0$$
&c.

And consequently,

$$p+q+r=0$$
$$p^2+q^2+r^2=0$$
$$p^3+q^3+r^3=3$$
$$p^4+q^4+r^4=0,\ \&c.$$

Make, now, $p^3=1$, $q^3=1$, and $r^3=1$; that is, let p, q, and r, be the three roots of the cubic equation $z^3=1$, or $z^3-1=0$: then, seeing both the second and third terms of this equation are wanting, not

not only the ſum of all the roots $(p+q+r)$ but the ſum of all their ſquares $(p^2+q^2+r^2)$ will vaniſh, or be equal to nothing (by common algebra), as they ought, to fulfil the conditions of the two firſt equations. Moreover, ſince $p^3=1$, $q^3=1$, and $r^3=1$, it is alſo evident, that $p^4+q^4+r^4$ $(=p+q+r)=0$, $p^5+q^5+r^5$ $(=p^2+q^2+r^2)=0$, $p^6+q^6+r^6$ $(=p^3+q^3+r^3)=3$. Which equations being, in effect, nothing more than the firſt three repeated, the values of p, q, r, above aſſigned, equally fulfil the conditions of theſe alſo: ſo that the ſeries ariſing from the addition of three aſſumed ones will agree, in every term, with *that* whoſe ſum is required: but thoſe ſeries (whereof the quantity in queſtion is compoſed) having all of them the *ſame form* and the *ſame cofficients* with the original ſeries $a+bx+cx^2+dx^3$, &c. $(=S)$, their ſums will therefore be truly obtained, by ſubſtituting px, qx, and rx, ſucceſſively, for x, in the given value of S. And, by the very ſame reaſoning, and the proceſs above laid down, it is evident, that, if every n^{th} term (inſtead of every third term) of the given ſeries be taken, the values of p, q, r, s, &c. will then be the roots of the equation $z^n-1=0$ *; and that, the

* If α, β, γ, δ, &c. be ſuppoſed to repreſent the co-ſines of the angles $\frac{360^\circ}{n}$, $2\times\frac{360^\circ}{n}$, $3\times\frac{360^\circ}{n}$, &c. (the radius being unity); then the roots of the equation $z^n-1=0$ (expreſſing the ſeveral values of p, q, r, s, &c.) will be truly defined by 1, $\alpha+\sqrt{\alpha\alpha-1}$, $\alpha-\sqrt{\alpha\alpha-1}$, $\beta+\sqrt{\beta\beta-1}$, $\beta-\sqrt{\beta\beta-1}$, &c. The demonſtration of this will be given farther on.

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ſum of all the terms ſo taken, will be truly obtained by ſubſtituting px, qx, rx, sx, &c. ſucceſſively for x, in the given value of S, and then dividing the ſum of all the quantities thence ariſing by the given number n.

The ſame method of ſolution holds equally, when, in taking every n^{th} term of the ſeries, the operation begins at ſome term after the firſt. For all the terms preceding *that* may be tranſpoſed, and the whole equation divided by the power of x in the firſt of the remaining terms; and then the ſum of every n^{th} term (beginning at the firſt) will be found by the preceding directions; which ſum, multiplied by the power of x that before divided, will evidently give the true value required to be determined. Thus, for example, let it be required to find the ſum of every third term of the given ſeries $a + bx + cx^2 + dx^3 + ex^4$, &c. $(= S)$, beginning with cx^2. Then, by tranſpoſing the two firſt terms, and dividing the whole by x^2, we ſhall have $c + dx + ex^2 + fx^3$, &c. $= \frac{S - a - bx}{xx}$ $(= S')$. From whence having found the ſum of every third term of the ſeries $c + dx + ex^2 + fx^3$, &c. beginning at the firſt (c), that ſum, multiplied by x^2, will manifeſtly give the true value ſought in the preſent caſe.

And here it may be worth while to obſerve, that all the terms preceding *that* at which the operation (in any caſe) begins, may (provided they exceed not in number the given interval n) be intirely diſ-

regarded,

regarded, as having no effect at all in the result. For if in that part $\left(\frac{-a-bx}{xx}\right)$ of the value of S', above exhibited, in which the first terms, a and bx, enter, there be substituted px, qx, rx, successively, for x (according to the *prescript)* the sum of the quantities thence arising will be

$$-\frac{a}{p^2x^2}-\frac{a}{q^2x^2}-\frac{a}{r^2x^2}$$
$$-\frac{b}{px}-\frac{b}{qx}-\frac{b}{rx}$$

which, because $p^3=1$, $q^3=1$, &c. (or $p^2=\frac{1}{p}$, $q^2=\frac{1}{q}$, &c.) may be expressed thus;

$$-\frac{a}{xx}\times\overline{p+q+r}$$
$$-\frac{b}{x}\times\overline{p^2+q^2+r^2}$$

But, that $p+q+r=0$, and $p^2+q^2+r^2=0$, hath been already shewn; whence the truth of the general observation is manifest. Hence it also appears, that the method of solution above delivered, is not only general, but includes this singular beauty and advantage, that in all series' whatever, whereof the terms are to be taken according to the same assigned order, the quantities $(p, q, r$, &c.), whereby the resolution is performed, will remain invariably the same. The greater part of these quantities are indeed *imaginary* ones; and so likewise will the quantities be that result from them, when substitution is made in the given expression for the value of S. But by adding, as is usual in like cases, every two corresponding values,

lues, ſo reſulting together, all marks of *impoſſibility* will diſappear.

If, in the ſeries to be ſummed, the alternate terms (*viz.* the 2d, 4th, 6th, *&c.*) ſhould be required to be taken under ſigns contrary to what they have in the original ſeries given; the reaſoning and reſult will be no-ways different; only, inſtead of making $p^3 + q^3 + r^3$ (or $p^n + q^n + r^n$, &c.) $= + 3$ (or $+ n$), the ſame quantity muſt, here, be made $= - 3$ (or $- n$). From whence, p^n being $= - 1$, $q^n = - 1$, &c. the values of p, q, r, &c. will, in this caſe, be the roots of the equation $z^n + 1 = 0$.

It may be proper, now, to put down an example, or two, of the uſe and application of the general concluſions above derived. Firſt, then, ſuppoſing the ſeries, whoſe ſum is given, to be $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \ldots\ldots + \frac{x^m}{m} + \frac{x^{m+1}}{m+1} + \frac{x^{m+2}}{m+2} \ldots\ldots + \frac{x^{m+n}}{m+n} + \frac{x^{m+n+1}}{m+n+1} +$, &c. $= -$ H. Log. $\overline{1 - x}$ $(= S)$; let it be required, from hence, to find the ſum of the ſeries $\left(\frac{x^m}{m} + \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n}\right.$ &c.) ariſing by taking every n^{th} term thereof, beginning with that whoſe exponent (m) is any integer leſs than n. Here, the terms preceding $\frac{x^m}{m}$ being tranſpoſed, and the whole equation divided by x^m; we we ſhall have $\frac{1}{m} + \frac{x}{m+1} + \frac{x^2}{m+2} + \frac{x^3}{m+3}$, &c. $= -\frac{1}{x^m} \times$ H. Log. $\overline{1-x} - \frac{x + \frac{1}{2}x^2, \&c.}{x^m}$. In which value, let px, qx, rx, &c. be, ſucceſſively, ſubſtituted for x (according to preſcript) neglecting intirely the terms $\frac{x + \frac{1}{2}x^2}{x^m}$, as having no effect at all in the reſult: from whence we get $-\frac{1}{\overline{px}|^m} \times$ Log. $\overline{1-px} - \frac{1}{\overline{qx}|^m} \times$ Log. $\overline{1-qx} - \frac{1}{\overline{rx}|^m} \times$ Log. $\overline{1-rx}$, &c. Which multiplied by x^m (the quantity that before divided) gives $-\frac{1}{p^m} \times$ Log. $\overline{1-px} - \frac{1}{q^m} \times$ Log. $\overline{1-qx} - \frac{1}{r^m} \times$ Log. $\overline{1-rx}$, &c. $=$ n times the quantity required to be determined.

But now, to get rid of the imaginary quantities q, r, &c. by means of their known values $\alpha + \sqrt{\alpha\alpha - 1}$, $\alpha - \sqrt{\alpha\alpha - 1}$, &c. it will be neceſſary to obſerve, that, as the product of any two correſponding ones ($\overline{\alpha + \sqrt{\alpha\alpha - 1}} \times \overline{\alpha - \sqrt{\alpha\alpha - 1}}$) is equal to unity, we may therefore write $\overline{\alpha - \sqrt{\alpha\alpha - 1}}|^m$ ($= r^m$) inſtead of its equal $\frac{1}{q^m}$, and $\overline{\alpha + \sqrt{\alpha\alpha - 1}}|^m$ ($= q^m$) inſtead of its equal $\frac{1}{r^m}$: by which means the two

 terms,

terms, wherein these two quantities enter, will stand thus; $-\overline{\alpha-\sqrt{\alpha\alpha-1}}|^{n}\times$ Log. $\overline{1-qx}$ $-\overline{\alpha+\sqrt{\alpha\alpha-1}}|^{m}\times$ Log. $\overline{1-rx}$.

But, if A be assumed to express the co-sine of an arch ($\mathfrak{Q}$), m times as great as that $\left(\frac{360^\circ}{n}\right)$ whose co-sine is here denoted by α; then will $A-\sqrt{AA-1}$ $=$ * $\overline{\alpha-\sqrt{\alpha\alpha-1}}|^{m}$, and $A+\sqrt{AA-1}=$

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* Because $\frac{-\dot{x}}{\sqrt{1-xx}}$ and $\frac{-\dot{X}}{\sqrt{1-XX}}$ are known to express the fluxions of the circular arcs whose co-sines are x and X, it is evident, if those arcs be supposed in any constant ratio of 1 to n, that $\frac{n\dot{x}}{\sqrt{1-xx}}=\frac{\dot{X}}{\sqrt{1-XX}}$, and consequently that $\frac{n\dot{x}}{\sqrt{xx-1}}$ $\left(=\frac{n\dot{x}}{\sqrt{-1}\times\sqrt{1-xx}}=\frac{\dot{X}}{\sqrt{-1}\times\sqrt{1-XX}}\right)=\frac{\dot{X}}{\sqrt{XX-1}}$. From whence, by taking the fluents, $n\times$ Log. $\overline{x+\sqrt{xx-1}}$ (or Log. $\overline{x+\sqrt{xx-1}}|^{n}$) $=$ Log. $\overline{X+\sqrt{XX-1}}$; and consequently $\overline{x+\sqrt{xx-1}}|^{n}=X+\sqrt{XX-1}$: whence also, seeing $x-\sqrt{xx-1}$ is the reciprocal of $x+\sqrt{xx-1}$, and $X-\sqrt{XX-1}$ of $X+\sqrt{XX-1}$, it is likewise evident, that $\overline{x-\sqrt{xx-1}}|^{n}=X-\sqrt{XX-1}$. Hence, not only the truth of the above assumption, but what has been advanced in relation to the roots of the equation $z^{n}-1=0$, will appear manifest. For if $x\pm\sqrt{xx-1}$ be put $=z$, then will z^{n} ($=\overline{x\pm\sqrt{xx-1}}|^{n}$) $=X\pm\sqrt{XX-1}$: where, assuming $X=1=$ co-s. $0=$ co-s. $360^\circ=$ co-s. $2\times360^\circ=$ co-s. $3\times360^\circ$, &c. the equation will become $z^{n}=1$, or $z^{n}-1=0$; and the different values of x, in the expression ($x\pm\sqrt{xx-1}$) for the root z, will consequently be the co-sines of the arcs, $\frac{0}{n}$, $\frac{360^\circ}{n}$, $\frac{2\times360^\circ}{n}$, &c. these arcs being

the

$\overline{\alpha + \sqrt{\alpha\alpha - 1}}|^m$: which values being fubftituted above, we thence get

$$-A \times \log. \overline{1 - qx} + \log. \overline{1 - rx}$$
$$+ \sqrt{AA - 1} \times \log. \overline{1 - qx} - \log. \overline{1 - rx};$$

whereof the former part (which, exclufive of the factor A, I fhall hereafter denote by M) is manifeftly equal to $-A \times \log. \overline{1 - qx} \times \overline{1 - rx}$ (by the nature of logarithms) $= -A \times \log. 1 - \overline{q + r}.x + qrx^2 = -A \times \log. \overline{1 - 2\alpha x + xx}$ (by fubftituting the values of q and r): which is now intirely free from imaginary quantities. But, in order to exterminate them out of the latter part alfo, put $y = \log. \overline{1 - qx} - \log. \overline{1 - rx}$; then will $\dot{y} = \frac{-q\dot{x}}{1 - qx} + \frac{r\dot{x}}{1 - rx} = -\frac{\overline{q - r} \times \dot{x}}{1 - \overline{q + r} \times x + xx} = -\frac{2\sqrt{\alpha\alpha - 1} \times \dot{x}}{1 - 2\alpha x + xx} = -\frac{2\sqrt{-1} \times \sqrt{1 - \alpha\alpha} \times \dot{x}}{1 - 2\alpha x + xx}$; where $\frac{\sqrt{1 - \alpha\alpha} \times \dot{x}}{1 - 2\alpha x + xx}$ expreffeth the fluxion of a circular arch (N) whofe radius is 1, and fine $= \frac{\sqrt{1 - \alpha\alpha} \times x}{1 - 2\alpha x + xx}$; confequently y will be $= -2\sqrt{-1} \times N$: which, multiplied by $\sqrt{AA - 1}$, or its equal $\sqrt{-1} \times \sqrt{1 - AA}$, gives $2\sqrt{1 - AA} \times N$;

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the correfponding *fubmultiples* of thofe above, anfwering to the cofine X ($= 1$). —— In the fame manner, if X be taken $= -1 =$ co-f. $180° =$ co-f. $3 \times 180° =$ co-f. $5 \times 180°$, &c. then will $z^n = -1$, or $z^n + 1 = 0$; and the values of x will, in this cafe, be the co-fines of $\frac{180°}{n}$, $3 \times \frac{180°}{n}$, $5 \times \frac{180°}{n}$, &c.

and,

and, this value being added to that of the former part (found above), and the whole being divided by n, we thence obtain $\frac{-AM + 2\sqrt{1-AA} \times N}{n}$, or $\frac{1}{n} \times \overline{-\text{co-s.}\, Q \times M + \text{sin.}\, Q \times 2N}$ for that part of the value sought depending on the two terms affected with q and r. From whence the sum of any other two corresponding terms will be had, by barely substituting one letter, or value, for another: So that,

$$\frac{1}{n} \times \begin{cases} -\log.\, \overline{1-x} \\ -\text{co-s.}\, Q \times M + \text{sin.}\, Q \times 2N \\ -\text{co-s.}\, Q' \times M' + \text{sin.}\, Q' \times 2N' \\ -\text{co-s.}\, Q'' \times M'' + \text{sin.}\, Q'' \times 2N'' \\ -\,\&c. \qquad\qquad + \&c. \end{cases}$$

will truly express the sum of the series proposed to be determined; M, M', M'' &c. being the hyperbolical logarithms of $1 - 2\alpha x + xx$, $1 - 2\beta x + xx$, $1 - 2\gamma x + xx$, &c. N, N', N'' &c. the arcs whose sines are $\frac{x\sqrt{1-\alpha\alpha}}{\sqrt{1-2\alpha x+xx}}$, $\frac{x\sqrt{1-\beta\beta}}{\sqrt{1-2\beta x+xx}}$, $\frac{x\sqrt{1-\gamma\gamma}}{\sqrt{1-2\gamma x+xx}}$, &c. and Q, Q', Q'' &c. the measures of the angles expressed by $\frac{360^\circ}{n} \times m$, $2 \times \frac{360}{n} \times m$, $3 \times \frac{360}{n} \times m$, &c. And here it may not be amiss to take notice, that the series $\frac{x^m}{m} + \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n} +$ &c. thus determined, is that expressing the fluent of $\frac{x^{m-1}\dot{x}}{1-x^n}$; corresponding to one of the two famous

Coteſian forms. From whence, and the reaſoning above laid down, the fluent of the other *form*, $\frac{x^{m-1}\dot{x}}{1+x^n}$, may be very readily deduced. For, ſince the ſeries $\left(\frac{x^m}{m} - \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n} - \frac{x^{m+3n}}{m+3n}\right.$ &c.) for this laſt fluent, is that which ariſes by changing the ſigns of the alternate terms of the former; the quantities p, q, r, &c. will here (agreeably to a preceding obſervation) be the roots of the equation $z^n + 1 = 0$; and, conſequently, α, β, γ, δ, &c. the co-ſines of the arcs $\frac{180°}{n}$, $3 \times \frac{180°}{n}$, $5 \times \frac{180°}{n}$, &c. (as appears by the foregoing note). So that, making Q, Q', Q'', &c. equal, here, to the meaſures of the angles $\frac{180°}{n} \times m$, $3 \times \frac{180°}{n} \times m$, $5 \times \frac{180°}{n} \times m$, &c. the fluent ſought will be expreſſed in the very ſame manner as in the preceding caſe; except that the firſt term, $-\log. \overline{1-x}$ (ariſing from the *rational* root $p = 1$) will here have no place.

After the ſame manner, with a ſmall increaſe of trouble, the fluent of $\frac{x^{m-1}\dot{x}}{1 \pm 2lx^n + x^{2n}}$ may be derived, m and n being any integers whatever. But I ſhall now put down one example, wherein the impoſſible quantities become exponents of the powers, in the terms where they are concerned.

The ſeries here given is $1 - x + \frac{x^2}{2} - \frac{x^3}{2.3} + \frac{x^4}{2.3.4} - \frac{x^5}{2.3.4.5}$, &c. = the number whoſe hyp. log. is is $-x$, and it is required to find the ſum of every n^{th} term thereof, beginning at the firſt. Here the quantity ſought will (according to the general rule) be truly defined by the n^{th} part of the ſum of all the numbers whoſe reſpective logarithms are $-px$, $-qx$, $-rx$, &c.; which numbers, if N be taken to denote the number whoſe hyp. log. $= 1$, will be truly expreſſed by N^{-px}, N^{-qx}, N^{-rx}, &c. From whence, by writing for p, q, r, &c. their equals $1, \alpha + \sqrt{\alpha\alpha - 1}, \alpha - \sqrt{\alpha\alpha - 1}, \beta + \sqrt{\beta\beta - 1}$, $\beta - \sqrt{\beta\beta - 1}$, &c. and putting $\acute{\alpha} = \sqrt{1 - \alpha\alpha}$, $\beta' = \sqrt{1 - \beta\beta}$, &c. we ſhall have $\frac{1}{n} \times \overline{N^{-px} + N^{-qx} + N^{-rx}}$, &c. $= \frac{1}{n}$ into $N^{-x} + N^{-\alpha x} \times \overline{N^{-\acute{\alpha}x\sqrt{-1}} + N^{\acute{\alpha}x\sqrt{-1}}} + N^{-\beta x} \times \overline{N^{-\beta' x\sqrt{-1}} + N^{\beta' x\sqrt{-1}}}$ + &c. But $N^{-\acute{\alpha}x\sqrt{-1}} + N^{\acute{\alpha}x\sqrt{-1}}$ is known to expreſs the double of the co-ſine of the arch whoſe meaſure (to the radius 1) is $\acute{\alpha}x$. Therefore we have $\frac{1}{n}$ into $N^{-x} + N^{-\alpha x} \times 2$ co-ſ. $\acute{\alpha}x$ + $N^{-\beta x} \times 2$ co-ſ. $\beta' x$, &c. for the true ſum, or value propoſed to be determined.

The ſoluion of this caſe, in a manner a little different, I have given ſome time ſince, in another place; where the principles of the general method, here extended and illuſtrated, are pointed out. I ſhall put an end to this paper with obſerving, that if, in the

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series given, the even powers of x, or any other terms whatever, be wanting, their places must be supplied with cyphers; which, in order the of numbering off, must be reckoned as real terms.

CIV. *Observatio Eclipsis Lunæ Die* 30 Julii 1757. *habita Olissipone à* Joanne Chevalier, *Congregationis Oratorii Presbytero, é Regia* Londinensi *Societate. Communicated by* Jacob de Castro Sarmiento, *M D F. R. S.*

Tubo optico 8 pedum.

Read Nov. 16. 1758.

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Initium penumbræ ——	9	15	18
Initium dubium eclipsis	9	22	24
Certo jam incœperat —— ——	9	23	34
Umbra ad mare humorum observata vitro plano cæruleo —— ——	9	31	2
Solo tubo optico observata —— ——	9	31	29
Vitro flavo observata —— ——	9	31	48
Umbra tangit Grimaldum observata vitro plano cæruleo —— ——	9	31	20
Solo tubo optico —— ——	9	31	50
Vitro plano flavo —— ——	9	32	8
Totus Grimaldus tegitur observatus vitro plano cæruleo —— ——	9	34	4
Solo tubo optico —— —— ——	9	34	28
Vitro flavo —— —— ——	9	34	47

Umbra